Sensitivity error bounds for non-exponental stochastic networks

van Dijk, N.M.

1989

SERIE RESEARCH MEMORANDA

SENSITIVITY ERROR BOUNDS FOR NON-EXPONENTIAL STOCHASTIC NETWORKS

by

Nico M. van Dijk

Research Memorandum 1989-93

December 1989

VRIJE UNIVERSITEIT
Faculteit der Economische Wetenschappen en Econometrie
A M S T E R D A M

SENSITIVITY ERROR BOUNDS FOR NON-EXPONENTIAL STOCHASTIC NETWORKS

Nico M. van Dijk
Free University, The Netherlands

Abstract Stochastic service networks are studied with inaccuracies or perturbations in the distributional forms of service and interarrival times. A condition is provided to conclude error bounds for the effect of these data imprecisions on stationary measures such as a throughput. The verification of this condition is illustrated for a Jacksonian network application with perturbed nonexponential input.

Keywords Stochastic network * performance measures * sensitivity error bound * Markov chain.

1 Introduction

Stochastic networks have gained a wide popularity over the last decades in telecommunication, computer performance evaluation and flexible manufacturing. Most notably, explicit product form expressions and related insensitivity properties have been intensively investigated (cf. [3, 4, 11, 14, 40]. Generally, these expressions rely upon assumptions such as Poissonian arrivals, exponential services, special service disciplines (e.g. processor sharing) and reversible or state independent routing. Such assumptions are typically not met in practice. Simulation, numerical or approximative techniques are then to be used. As these techniques can be computationally expensive, robustness or sensitivity results with respect to system input data are of interest.

Particularly, the distributional forms of interarrival and service times are a key-factor for both computational and sensitivity results. As these forms are usually obtained by experimental data inaccuracies are naturally involved. Error bound results on the effect of distributional imprecisions are thus of natural interest. Conversely, for simulation or approximation purposes robustness results may provide one flexibility in choosing convenient distributional forms (for example Weibull (easily invertible) or Erlang (Markov chain analysis)).

Perturbation or sensitivity analysis has recently enjoyed considerable attention in connection with simulation (cf. [5, 6, 7, 8, 9, 10, 24 25]). This analysis, however, provides sensitivity bounds based on simulated sample path outcomes and does not secure a priori bounds. Moreover, only a number of system parameters and not total distributional forms are hereby studied. Analytic perturbation or related truncation results that provide one a priori error bounds for stationary characteristics have also been obtained (e.g. [12, 13, 15, 16, 17, 18, 29, 35]). Without exception though these are all concerned with discrete state Markov chains, while no analogues have been reported for continuous state Markov chains as the present paper requires. Alternatively, for particular examples insensitive bounds have been established by modifying the original system in insensitive product forms systems (cf. [28, 30, 31]). This technique, however, is limited to special systems and does not apply to interarrival times. Moreover, these bounds are merely first quick indicators and do not secure orders of accuracy for small data perturbations.

This paper provides an analytical tool to conclude explicit a priori error bounds for the amount of sensitivity due to imprecisions in distributional data for interarrival and service times. The essential step to this end is the estimation of so-called bias terms of an underlying reward structure. Such estimates have recently been established for various discrete state queueing network applications, cf. [31, 32, 34, 36]. The present application though requires a continuous state description and bias-terms estimates in that case do not seem to be available. To illustrate how this can be established analytically in concrete multi-dimensional situations, a Jacksonnian network with a nonexponential renewal input is studied in detail.

2 General model

For presentational convenience we first consider the closed case. The open case will be illustrated by the Jacksonian application.

Consider a closed stochastic network with a fixed number of M jobs, numbered 1,...,M. At any moment the state of the system is represented by

$$(L,T)$$

where

$$L = (\alpha_1,\ldots,\alpha_M)$$

$$T = (t_1,\ldots,t_M)$$

denotes for each job i its current status $\alpha_i \in S$, with S some countable set and time (age) t_i after the last service completion of this job.

Example (Queueing Network) In a queueing network we can have: $\alpha_i=(r,j,p)$ denoting for job i its current type number r, the station j at which it is present and the queueing position p while t_i is the time that it has already been present at this station.

Law of motion The system dynamics are determined by the system characteristics

$F_\alpha(.)$	: distribution functions
$s_i([L,T])$	: service rates (speeds)
$p_i(\alpha \mid [L,T])$	: transition probabilities

as follows. When a job changes its jobmark in ℓ it requires a random amount of service with distribution function F_ℓ. When the system is in state $[L,T]$, the service rate i.e. the amount of service per unit of time provided to job i is $s_i([L,T])$. When the system is in state $[L,T]$ and job i completes its service, its jobmark is changed in α' with probability $p_i(\alpha' \mid [L,T])$.

Remarks

1. Note that the service rate for a particular job in a particular state can be equal to zero. This naturally arises for instance in a queueing network with FCFS-service stations, as will be illustrated in the example below.
2. Clearly, the above parametrization could have been combined in one service completion rate function. However, the present more detailed formulation is preferred as it corresponds more naturally to queueing network protocols.

Example (Queueing Network) Consider a closed queuing network with N first-come first-served (FCFS)-single server service stations and M numbered jobs. A job requires random amounts of service at the various stations, say at station j according to a distribution function G_j. The service rate at station j is $s_j(n_j,t_j)$ when n_j jobs are present while the job in service has received already t_j units of service. (The service rate thus depends on the total amount of residual workload at this station). Upon service completion at station j a job routes to station k with probability $p_{jk},(\bar{n},\bar{t})$ where $\bar{n} = (n_1,\ldots,n_N)$ and $\bar{t} = (t_1,\ldots,t_N)$. (Age and workload dependent routing as well as blocking are hereby involved).

Let $\alpha = (i,j,p)$ denote the job-number i of a job, the station number j at which it is present and the position p at this station that it occupies, where p = 1 represents the head of the queue. Also, read $1(A) = 1$ if an event A is satisfied and $1(A) = 0$ if not. The above parametrization then applies with

$$F_\alpha = G_j$$

$$s_i([L,T]) = s_j(n_j,t_j)\ 1(p=1)$$

$$p_i(\alpha'\,|\,[L,T]) = p_{jk},(\bar{n},\bar{t})\ 1(\alpha'=(i,k,p)) \qquad \square$$

The following assumptions are made in order to define a convenient transformation.

Assumptions

1. For all α, the function $F_\alpha(t)$ is absolute continuous for $t\in(0,\infty)$ with density function $f_\alpha(t)$. Hence, its failure rate is well-defined by $f_\alpha(t)/[1-F_\alpha(t)]$ for all $t\in(0,\infty)$. We introduce the notation:

(1) $\quad d_i([L,T]) = s_i([L,T])\ f_{\alpha_i}(t_i)/[1-F_{\alpha_i}(t_i)]$

2. For some constant $B < \infty$ and all $[L,T]$:

(2) $\quad d([L,T]) = \Sigma_i\ d_i([L,T]) \le B$

Uniformized model We now define a related continuous-time Markov chain model as follows. At exponential times with parameter Q the system will have a jump. For $v > 0$ let T+v denote the vector $(t_1+v,t_2+v,\ldots,t_M+v)$. If directly after the last jump the system was in state [L,T] while the next jump will take place after time v, by this next jump with probability

(3) $\quad p_v([L,T],\ [L',T']) = d_i([L,T+v])\ p_i(\alpha'\,|\,[L,T+v])/Q$

the system will change in state [L',T'] with $\alpha_j'=\alpha_j$ and $t_j=t_j+v$ for $j\neq i$ but $\alpha_i'=\alpha'$ and $t_i'=0$, for all $i=1,\ldots,M$, while with probability

(4) $\quad 1-d([L,T+v])/Q$

only the ages are updated, i.e. the state will change in [L,T+v].

Without restriction of generality, assume that both the original and the above uniformized model have a unique density stationary distribution at one and the same irreducible set R which we denote by $\pi_1(L,T)$ and $\pi_2(L,T)$ respectively. The following lemma is proven in [36].

Lemma 1.1

$$\pi_1(.) = \pi_2(.).$$

2 Sensitivity bounds

Consider a similar perturbed stochastic network with the characteristics

$$F_\alpha(.),\ s_i(.),\ p_i(.|.)$$

replaced by

$$\bar{F}_\alpha(.),\ \bar{s}_i(.),\ \bar{p}_i(.|.)$$

And without restriction of generality assume that (1) and (2) hold again with the same value Q. Then by virtue of lemma 1, its stationary distribution, denoted by $\bar{\pi}(.)$, is also determined by the uniformized model with the above substitutions. From now on, we will always denote an expression for this perturbed system with an upper bar symbol '-', while the symbol '(-)'is used when both the original and perturbed model are meant.

Let R(L,T) be some reward function and define functions $\bar{V}_n(.)$ and $V_n(.)$ for n=0,1,2... by $\bar{V}_0(.)=V_0(.)=0$ and for n=0,1,2,...:

$$(5)\quad \overset{(-)}{V}_{n+1}(L,T) = \overset{(-)}{R}(L,T)+{}_0\!\int^\infty Qe^{-vQ}\Sigma_{[L',T']}\overset{(-)}{p}_v([L,T],[L',T'])\overset{(-)}{V}_n(L',T')dv$$

where it is to be noted that the summation over [L',T'] for fixed value v actually comes down to summation over all possible components i which determines which component will change or, [L',T']=[L,T+v]. In words that is, $V_n(L,T)$ represents the total expected reward over n exponential periods with parameter Q under the one-step transition structure $p_v(.,.)$ and one-step rewards R(.,.) and given the initial state [L,T] at time 0. Now assume that for some initial state $[L_0,T_0]$

$$(6)\quad \overset{(-)}{g} = \lim_{N\to\infty} \frac{Q}{N}\overset{(-)}{V}_N(L_0,T_0)$$

exists and is well-defined. As R(.) represents a one-step reward per period of expected length Q^{-1}, the values $\overset{(-)}{g}$ then represent an expected reward per unit of time when the system is in equilibrium. For example, for some given reward rate function $\overset{(-)}{r}(L,T)$ we can have

$$(7)\quad \overset{(-)}{R}(L,T) = {}_0\!\int^\infty Qe^{-vQ}\overset{(-)}{r}(L,T+v)dv$$

as corresponding to a reward measurement just prior to jumps, or

$$\overset{(-)}{R}(L,T) = {}_0\!\int^\infty Qe^{-vQ}\ [{}_0\!\int^v \overset{(-)}{r}(L,T+s)ds]dv$$

as a reward rate measurement continuously in time. The following keytheorem can now be formulated. Herein, let $[\bar{P}-P]([.,.],[.,.'])$ = $\bar{P}([.,.],[.,.'])-P([.,.],[.,.'])$.

Theorem 2.1 Suppose that for some $\Delta_1,\Delta_2 \geq 0$ and all $[L,T]$, $n \geq 0$:

(8) $$\left|\Sigma_{[L',T']}[\bar{P}-P]([L,T],[L',T'])\ [V_n(L',T')-V_n(L,T)]\right| \leq \Delta_1/Q$$

(9) $$|\bar{R}(L,T)-R(L,T)| \leq \Delta_2/Q.$$

Then

(10) $$|\bar{g}-g| \leq \Delta_1+\Delta_2.$$

Proof By virtue of (6):

(11) $$(\bar{V}_{n+1}-V_{n+1})([L,T]) = [\bar{R}(L,T) - R(L,T)] +$$

$$\int_0^\infty Qe^{-vQ}\left\{\sum_{[L',T']}\left[\bar{p}_v([L,T],[L',T'])-p_v([L,T],[L',T'])\right]V_n(L',T') + \sum_{[L',T']}\bar{p}_v([L,T],[L',T'])[\bar{V}_n(L',T') - V_n(L',T')]\right\}dv$$

Noting that

$$\sum_{[L',T']}\overset{(-)}{p}_v([L,T],\ [L',T']) = 1,$$

we have

(12) $$\sum_{[L',T']}\left[\bar{p}_v([L,T],[L',T']) - p_v([L,T],[L',T'])\right]\ V_n(L',T') =$$

$$\sum_{[L',T']}\left[\bar{p}_v([L,T],[L',T']) - p_v([L,T],[L',T'])\right] \times$$

$$[V_n(L',T')-V_n(L,T)].$$

Substituting (12) in (11) and applying (8) yields for any $[L,T]$:

$$\gamma_{n+1} = \sup_{[L,T]}|\bar{V}_{n+1}(L,T)-V_{n+1}(L,T)| \leq$$

$$\int_0^\infty \Delta_1 e^{-vQ}\ dv + \Delta_2 Q^{-1} + \sup_{[L',T']}[\bar{V}_n(L',T')-V_n(L',T')] \leq \gamma_n + [\Delta_1+\Delta_2]Q^{-1}.$$

Iterating this expression for $n=N-1,\ldots,0$ and substituting $\bar{V}_0(.)$ =

$V_0(.) = 0$ gives for any $[L,T]$:

$$|\bar{V}_N(L,T) - V_N(L,T)| \leq [N/Q][\Delta_1+\Delta_2].$$

Inserting $[L,T]=[L_0,T_0]$ and applying (7) completes the proof. □

Corollary 2.1 Let

$$\bar{R}(.) = R(.)$$

$$\bar{s}_i(.) = s_i(.) \leq S$$

(13)
$$\bar{p}_i(.|.) = p_i(.|.)$$

$$\overset{(-)}{h}_\alpha(t) = \overset{(-)}{f}_\alpha(.)/[1-\overset{(-)}{F}_\alpha(t)]$$

and assume that for some constants δ and $C \geq 0$:

(14) $|\bar{h}_\alpha(t) - h_\alpha(t)| \leq \delta$

(15) $|V_n(L',T') - V_n(L,T)| \leq C$

for all α and $n,t \geq 0$ and $[L',T']$ with $(\alpha_j', t_j') = (\alpha_j,t_j)$ for $j \neq i$ while $t_i'=0$ for some $i \in \{1,\ldots,n\}$. Then

(16) $|\bar{g}-g| \leq \delta CS$ □

Remark 2.1 (Monotonicity results) The proof can almost reread identically to conclude monotonicity results of the form

$$\bar{g} \leq g \quad \text{or} \quad \bar{g} \geq g$$

when (8) holds without absolute values and the right hand side replaced by ≤ 0 or ≥ 0. Monotonicity results for queueing networks have been extensively studied over the last decade such as with respect to the number of servers or jobs. (e.g. [1, 2, 19, 20, 21, 22, 23, 26, 28, 36, 38, 39]). Monotonicity results with respect to distributional forms though are limited to some results for simple Erlang type facilities (cf. [22, 38]). As such, the above result in monotonicity form would be an extended form of possible interest. The primary focus herein however are error bounds.

3 Application: A Jackson network with non-exponential input

To illustrate how the necessary condition (8) or (15) can be verified in concrete situations, this section investigatese a particular application: A finite tandem line. As a non exponential input is a realistic phenomenon but also known to be a key-factor for the failure of an explicit product form result, we particularize this application to a non-exponential input while for convenience of presentation services are assumed to be exponential.

The system under study is an open two station tandem line with a finite capacity constraint of no more than N jobs. Jobs arrive at the system according to a renewal input with interarrival (renewal) distribution F(.). When the system is saturated, i.e. n=N where n is the number of jobs already present, an arriving job is rejected and lost. Otherwise it enters station 1. After service completion at station 1 a job instantly routes to station 2 and after service completion at station 2 it directly leaves the system. When n_i jobs are present at station i the rate at which jobs are completed is $\mu_i(n_i)$, where $\mu_i(n_i)$ is assumed to be non-decreasing, i=1,2. (Services are thus assumed to be exponential).

The state of the system can be described by $[\bar{n},t]$ where $\bar{n}=(n_1,n_2)$ denotes the number n_i at station i=1,2 and where t is the time after the last arrival. The results of section 2 do not apply directly as the number of jobs is not fixed and external arrivals are involved. A way to include open models in the description of section 2 is to let arriving jobs be assigned an arrival number, to be included in the status α of a job, and to use a special number 0 to describe an external job which can enter the system or which is created when a job leaves the system. However, for the special system under consideration, we prefer to give a somewhat more direct version. More precisely, consider the system described above but with the interarrival distribution modified in $\bar{F}(.)$ and let

$$\overset{(-)}{h}(t) = \overset{(-)}{f}(t)/[1 - \overset{(-)}{F}(t)]$$

be the corresponding arrival failure rates for the original and modified system, which are assumed to be well-defined for all $t\in(0,\infty)$.
Now, with

$$\overset{(-)}{r}(\bar{n},T+v) = \overset{(-)}{h}(t+v)/Q$$

and

$$(17) \qquad Q \geq \sup_t 2\overset{(-)}{h}(t) + \sup_{\bar{n}}[\mu_1(n_1) + \mu_2(n_2)],$$

for Z=0,1,2,... define functions $\overset{(-)}{V}_Z(\bar{n},t)$ as per (5) and (7).
More precisely, define $V_0(.)=0$ and

$$(18)\qquad \overset{(-)}{V}_{m+1}(\bar{n},t) =$$

$$_0\int^{\infty} e^{-vQ}\left\{\overset{(-)}{h}(t+v)\, 1_{(n<N)} + \overset{(-)}{h}(t+v)\,\overset{(-)}{V}_m(\bar{n}+e_1,0) + \right.$$

$$[\mu_1(n_1)\,\overset{(-)}{V}_m\,(\bar{n}-e_1+e_2,t+v)+\mu_2(n_2)\,\overset{(-)}{V}_m(\bar{n}-e_2,t+v)] +$$

$$\left.[Q - \overset{(-)}{h}(t+v) - \mu_1(n_1) - \mu_2(n_2)]\,\overset{(-)}{V}_m(\bar{n},t+v)\right\} dv.$$

The values $\overset{(-)}{g}$ as defined by

$$(19)\qquad \overset{(-)}{g} = \lim_{Z\to\infty} \frac{Q}{Z}\,\overset{(-)}{V}_Z\,(\bar{0},0),$$

where $\bar{0}=(0,0)$, then represent the total system throughput, that is the mean number of accepted jobs or system departures per unit of time, when the system is in equilibrium. Now similarly to theorem 2.1 and using the fact that the systems differ in only their arrival failure rates, we can prove

Result 3.1

$$(20)\qquad |g - \bar{g}| \le \delta[1+C]$$

when for all $\bar{n}+e_1$ and $m,t\ge 0$):

$$(21)\qquad |h(t) - \bar{h}(t)| \le \delta$$

$$(22)\qquad |V_m(\bar{n}+e_1,0) - V_m(\bar{n},t)| \le C$$

As condition (21) is determined by the system data or modeling, the essential condition to be verified is (22). The following result proves the concrete simple estimate C=1 when the arrival failure rate is monotone nondecreasing.

Result 3.2. Assume that $h(t)$ is nondecreasing in t. Then for all $\bar{n}$, t, s, i and z:

$$(23)\qquad 0 \le \delta_s V_z(\bar{n},t) = V_z(\bar{n},t+s) - V_z(\bar{n},t) \le 1$$

$$(24)\qquad 0 \ge \Delta_1^s V_z(\bar{n},t) = V_z(\bar{n}+e_1,t) - V_z(\bar{n},t+s) \ge -1$$

$$(25)\qquad 0 \ge \Delta_2^s V_z(\bar{n},t) = V_z(\bar{n}+e_2,t) - V_z(\bar{n},t+s) \ge -1$$

$$(26)\qquad 0 \ge \Delta_3^s V_z(\bar{n},t) = V_z(\bar{n}+e_1,t) - V_z(\bar{n}+e_2,t+s) \ge -1$$

Proof. This will be given by induction to z. Clearly, (23)-(26) hold for $z=0$ as $V_0(.)=0$. Suppose that (23)-(26) hold for $z=m$. Below we will then express V_{m+1} in V_m by means of (18). Before doing so it is noted in advance that in the derivations that will follow some terms are artificially added and substracted (e.g. $h(t+s+v)-h(t+v)$ in (27) and $\mu_1(n_1+1)-\mu_1(n_1)$ in (28)) or artifically split (e.g. $h(t+s+v)$ in $h(t+v)+[h(t+s+v)-h(t+v)]$ in (27) and $\mu_i(n_i+1)=\mu_i(n_i)+[\mu_i(n_i+1)-\mu_i(n_i)]$ in (28)) in order to compare corresponding terms pairwise with equal coefficients. Further, as the detailed technicalities are slightly different but crucial, the derivations will be given in full detail for all inequalities to be proven.

$$
\begin{aligned}
(27)\qquad & \delta_s V_{m+1}(\bar{n},t) = \\
= & \int_0^\infty e^{-vQ}\Big\{h(t+s+v)\,1_{\{n<N\}} + \\
& \quad h(t+v)\,1_{\{n<N\}}\,V_m(\bar{n}+e_1,o) + \\
& \quad [h(t+s+v)-h(t+v)]\,1_{\{n<N\}}V_m(\bar{n}+e_1,o) + \\
& \quad \mu_1(n_1)V_m(\bar{n}-e_1+e_2,t+s+v) + \mu_2(n_2)V_m(\bar{n}-e_2,t+s+v) + \\
& \quad [Q-h(t+s+v)1_{\{n<N\}} - \mu_1(n_1) - \mu_2(n_2)]\,V_m(\bar{n},t+s+v)\Big\}\,dv \\
- & \\
& \int_0^\infty e^{-vQ}\Big\{h(t+v)\,1_{\{n<N\}} + \\
& \quad h(t+v)\,1_{\{n<N\}}\,V_m(\bar{n}+e_1,o) + \\
& \quad [h(t+s+v)-h(t+v)]1_{\{n<N\}}\,V_m(\bar{n},t+v) + \\
& \quad \mu_1(n_1)V_m(\bar{n}-e_1+e_2,t+s+v) + \mu_2(n_2)V_m(\bar{n}-e_2,t+v) + \\
& \quad [Q-h(t+s+v)1_{\{n<N\}} - \mu_1(n_1) - \mu_2(n_2)]\,V_m(\bar{n},t+v)\Big\}\,dv \\
= & \\
& \int_0^\infty e^{-vQ}\Big\{[h(t+s+v)-h(t+v)]\,1_{\{n<N\}} + \\
& \quad [h(t+s+v)-h(t+v)]\,1_{\{n<N\}}\,\Delta_1^{[t+v]}V_m(\bar{n},0) + \\
& \quad \mu_1(n_1)\,\delta_s V_m(\bar{n}-e_1+e_2,t+v) + \mu_2(n_2)\,\delta_s V_m(\bar{n}-e_2,t+v) + \\
& \quad [Q-h(t+s+v)\,1_{\{n<N\}} - \mu_1(n_1) - \mu_2(n_2)]\,\delta_s V_m(\bar{n},t+v))\Big\}\,dv.
\end{aligned}
$$

Now note that by induction hypothesis (24) for $z=m$, the second term between braces $\{.\}$ in the latter expression can be negative but is estimated from below by $-[h(t+s+v)-h(t+v)]\,1_{\{n<N\}}$. By combining this negative estimate with the first positive term, which is exactly the same up to sign, applying the induction hypothesis $\delta_s V_m(.)\geq 0$ and recalling (23), one concludes: $\delta_s V_m(.)\geq 0$. To estimate the latter expression from above by 1, delete the second term which is nonpositive by induction assumption, apply the induction hypothesis $\delta_s V_m(.)\leq 1$ and note that all terms between braces $\{.\}$ now sum up to 1 by virtue of (17). Inequality (23) is hereby verified for $z=m+1$.

To verify (24), again we will apply (18) where the remarks made above are recalled. We then obtain:

(28) $\Delta_1^s V_m(\bar{n},t)$

$$=$$

$$\int_0^\infty e^{-vQ} \Big\{ h(t+v)\, 1_{\{n+1<N\}} +$$

$$h(t+v)\, 1_{\{n+1<N\}}\, V_m(\bar{n}+e_1+e_1,0) +$$

$$[h(t+s+v)-h(t+v)]\, 1_{\{n+1<N\}} V_m(\bar{n}+e_1,t+v) +$$

$$h(t+v)\, 1_{\{n+1=N\}} V_m(\bar{n}+e_1,t+v) +$$

$$[h(t+s+v)-h(t+v)]\, 1_{\{n+1=N\}} V_m(\bar{n}+e_1,t+v) +$$

$$[\mu_1(n_1+1) - \mu_1(n_1)]\, V_m(\bar{n}+e_2,t+v) +$$

$$\mu_1(n_1)\, V_m(\bar{n}+e_2,t+v) + \mu_2(n_2)\, V_m(\bar{n}+e_1-e_2,t+v) +$$

$$[Q-h(t+s+v)-\mu_1(n_1+1)-\mu_2(n_2)] V_m(\bar{n}+e_1,t+v) \Big\}\, dv$$

$$-$$

$$\int_0^\infty e^{-vQ} \Big\{ h(t+s+v)\, 1_{\{n+1<N\}} + h(t+s+v)\, 1_{\{n+1=N\}} +$$

$$h(t+v)\, 1_{\{n+1<N\}}\, V_m(\bar{n}+e_1,0) +$$

$$h(t+v)\, 1_{\{n+1=N\}}\, V_m(\bar{n}+e_1,0) +$$

$$[h(t+s+v)-h(t+v)]\, 1_{\{n+1<N\}}\, V_m(\bar{n}+e_1,0) +$$

$$[h(t+s+v)-h(t+v)]\, 1_{\{n+1=N\}}\, V_m(\bar{n}+e_1,0) +$$

$$[\mu_1(n_1+1)-\mu_1(n_1)]\, V_m(\bar{n},t+s+v) +$$

$$\mu_1(n_1)\, V_m(\bar{n}-e_1+e_2,t+s+v) + \mu_2(n_2)\, V_m(\bar{n}-e_2,t+s+v) +$$

$$[Q-h(t+s+v)-\mu_1(n_1+1)-\mu_2(n_2)]\, V_m(\bar{n},t+s+v) \Big\}\, dv$$

$$=$$

$$\int_0^\infty e^{-vQ} \Big\{ [h(t+v)-h(t+s+v)]\, 1_{\{n+1<N\}} - h(t+s+v)\, 1_{\{n+1=N\}} +$$

$$h(t+v)\, 1_{\{n+1<N\}}\, \Delta_1^0\, V_m(\bar{n},0) +$$

$$h(t+v)\, 1_{\{n+1=N\}}\, \delta_{\{t+v\}}\, V_m(\bar{n}+e_1,0) +$$

$$[h(t+s+v)-h(t+v)]\, 1_{\{n+1<N\}}\, \delta_{[t+v]}\, V_m(\bar{n}+e_1,0) +$$

$$[h(t+s+v)-h(t+v)]\, 1_{\{n+1=N\}}\, \delta_{[t+v]}\, \bar{V}_m(\bar{n}+e_1,0) +$$

$$[\mu_1(n_1+1)-\mu_1(n_1)]\ \Delta_2^s\ V_m(\bar{n},t+v)\ +$$
$$\mu_1(n_1)\ \Delta_1^s\ V_m(\bar{n}-e_1+e_2,t+v)\ +$$
$$\mu_2(n_2)\ \Delta_1^s\ V_m(\bar{n}-e_2,t+v)]\ +$$
$$\left.[Q-h(t+s+v)-\mu_1(n_1+1)-\mu_2(n_2)]\ \Delta_1^s\ V_m(\bar{n},t+v)\right\}\ dv.$$

Here note all $\delta_{[t+v]}\ V_m(.)$ are nonnegative by induction hypothesis (23) for z=m, but estimated from above by 1. As a consequence, by substituting these upper estimates, combining them with the first two negative terms, which are exactly equal to their coefficients up to sign, applying the induction hypotheses $\Delta_2^s V_m(.)\le 0$ and $\Delta_1^s\ V_m(.)\le 0$ and recalling (17), we conclude: $\Delta_1^s\ V_{m+1}(.)\le 0$. To estimate the latter expression from below by -1, delete all $\delta_{[t+v]}\ V_m(.)$-terms, which are nonnegative by induction assumption, apply the induction hypotheses $\Delta_2^s\ V_m(.)\ge -1$ and $\Delta_1^s\ V_m(.)\ge -1$ and note that all terms between braces {.} now sum up to -1 by virtue of (17). Inequality (24) is hereby verified for z=m+1.

To prove (25), we obtain similarly to (27):

(29) $\Delta_2^s\ V_m(\bar{n},t)$

$$=$$
$$\int_0^\infty e^{-vQ} \left\{ h(t+v)\ 1_{\{n+1<N\}}\ +\right.$$
$$h(t+v)\ 1_{\{n+1<N\}}\ V_m(\bar{n}+e_1+e_2,0)\ +$$
$$h(t+v)\ 1_{\{n+1=N\}}\ V_m(\bar{n}+e_2,t+v)\ +$$
$$[h(t+s+v)-h(t+v)]\ 1_{\{n+1<N\}}\ V_m(\bar{n}+e_2,t+v)\ +$$
$$[h(t+s+v)-h(t+v)]\ 1_{\{n+1=N\}}\ V_m(\bar{n}+e_2,t+v)\ +$$
$$[\mu_2(n_2+1)-\mu_2(n_2)]\ V_m(\bar{n},t+v)\ +$$
$$\mu_1(n_1)\ V_m(\bar{n}-e_1+e_2+e_2,t+v)\ +$$
$$\mu_2(n_2)\ V_m(\bar{n},t+v)\ +$$
$$\left.[Q-h(t+s+v)-\mu_1(n_1)-\mu_2(n_2+1)]\ V_m(\bar{n},t+v)\right\}\ dv$$

$$-$$

$$\int_0^\infty e^{-vQ} \left\{ h(t+s+v)\ 1_{\{n+1<N\}} + h(t+s+v)\ 1_{\{n+1=N\}}\ +\right.$$
$$h(t+v)\ 1_{\{n+1<N\}}\ V_m(\bar{n}+e_1,0)\ +$$
$$h(t+v)\ 1_{\{n+1=N\}}\ V_m(\bar{n}+e_1,0)\ +$$
$$[h(t+s+v)-h(t+v)]\ 1_{\{n+1<N\}}\ V_m(\bar{n}+e_1,0)\ +$$

$$[h(t+s+v)-h(t+v)]\ 1_{\{n+1=N\}}\ V_m(\bar{n}+e_1,0)\ +$$
$$[\mu_2(n_2+1)-\mu_2(n_2)]\ V_m(\bar{n},t+s+v)\ +$$
$$\mu_1(n_1)\ V_m(\bar{n}-e_1+e_2,t+s+v)\ +\ \mu_2(n_2)\ V_m(\bar{n}-e_2,t+s+v)\ +$$
$$\left.[Q-h(t+s+v)-\mu_1(n_1)-\mu_2(n_2+1)]\ V_m(\bar{n},t+s+v)\right\}\ dv$$

$=$

$$\int_0^\infty e^{-vQ}\ \Big\{[h(t+v)-h(t+s+v)]\ 1_{\{n+1<N\}}\ -\ h(t+s+v)\ 1_{\{n+1=N\}}\ +$$
$$h(t+v)\ 1_{\{n+1<N\}}\ \Delta_2^s V_m(\bar{n}+e_1,0)\ +$$
$$h(t+v)\ 1_{\{n+1=N\}}\ [-\Delta_3^{\{t+v\}} V_m(\bar{n},0)]\ +$$
$$[h(t+s+v)-h(t+v)]\ 1_{\{n+1<N\}}\ [-\Delta_3^{\{t+v\}} V_m(\bar{n},0)]\ +$$
$$[h(t+s+v)-h(t+v)]\ 1_{\{n+1=N\}}\ [-\Delta_3^{\{t+v\}} V_m(\bar{n},0)]\ +$$
$$[\mu_2(n_2+1)-\mu_2(n_2)][-\delta_s V_m(\bar{n},t+v)]\ +$$
$$\mu_1(n_1)\ \Delta_2^s V_m(\bar{n}-e_1+e_2,t+v)\ +$$
$$\mu_2(n_2)\ \Delta_2^s V_m(\bar{n}-e_2,t+v)\ +$$
$$\left.[Q-h(t+s+v)-\mu_1(n_1)-\mu_2(n_2+1)]\Delta_2^s V_m(\bar{n},t+v)\right\}\ dv.$$

Now note that all $-\Delta_3 V_m(.)$-terms are nonnegative as per induction hypothesis (26) for z=m but estimated from above by 1. Hence, as in (27) by substituting this upper estimates, combining them with the first two negative terms which are exactly equal to their coefficients up to sign applying the induction hypotheses $-\delta_s V_m(.)\le 0$ and $\Delta_2^s V_m(.)\le 0$ and recalling (17) we conclude $\Delta_2^s V_{m+1}(.)\le 0$.

Conversely, as before, by deleting the nonnegative $-\Delta_3^s V_m(.)$ terms applying $-\delta_s V_m(.)\ge -1$ and $\Delta_2^s V_m(.)\ge -1$ as per hypotheses and noting that all terms between braces then sum up to -1 by virtue of (17) we obtain $\Delta_2^s V_{m+1}(.)\ge -1$. Inequality (25) is thus proven for z=m+1.

Finally, again as in (28) we conclude:

$$(30)\quad \Delta_3^s V_m(\bar{n},t) =$$
$$\int_0^\infty e^{-vQ}\ \Big\{h(t+v)\ 1_{\{n+1<N\}}\ +$$
$$h(t+v)\ 1_{\{n+1<N\}} V_m(\bar{n}+e_1+e_1,0)\ +$$
$$[h(t+s+v)-h(t+v)]\ 1_{\{n+1<N\}} V_m(\bar{n}+e_1,t+v)\ +$$
$$[\mu_1(n_1+1)-\mu_1(n_1)]\ V_m(\bar{n}+e_2,t+v)\ +\ \mu_1(n_1) V_m(\bar{n}+e_2,t+v)\ +$$
$$[\mu_2(n_2+1)-\mu_2(n_2)]\ V_m(\bar{n}+e_1,t+v)\ +\ \mu_2(n_2) V_m(\bar{n}+e_1-e_2,t+v)\ +$$

$$[Q - h(t+s+v)\ 1_{\{n+1<N\}} - \mu_1(n_1+1) - \mu_2(n_2+1)]\ V_m(\bar{n}+e_1, t+v)\Big\}\ dv$$

$$= {}_0\!\int^{\infty} e^{-vQ} \Big\{h(t+s+v)\ 1_{\{n+1<N\}} +$$

$$h(t+v)\ 1_{\{n+1<N\}}\ V_m(\bar{n}+e_1+e_2, 0) +$$

$$[h(t+s+v) - h(t+v)]\ 1_{\{n+1<N\}}\ V_m(\bar{n}+e_1+e_2, 0) +$$

$$[\mu_1(n_1+1) - \mu_1(n_1)]\ V_m(\bar{n}+e_2, t+s+v) + \mu_1(n_1)\ V_m(\bar{n}-e_1+e_2+e_2, t+s+v) +$$

$$[\mu_2(n_2+1) - \mu_2(n_2)]\ V_m(\bar{n}, t+s+v) + \mu_2(n_2)\ V_m(\bar{n}, t+s+v) +$$

$$[Q - h(t+s+v)\ 1_{\{n+1<N\}} - \mu_1(n_1+1) - \mu_2(n_2+1)]\ V_m(\bar{n}+e_2, t+s+v)\Big\} dv$$

$$= {}_0\!\int^{\infty} e^{-vQ} \Big\{[h(t+v) - h(t+s+v)]\ 1_{\{n+1<N\}} +$$

$$h(t+v)\ 1_{\{n+1<N\}}\ \Delta_3^0 V_m(\bar{n}+e_1, 0) +$$

$$[h(t+s+v) - h(t+v)]\ 1_{\{n+1<N\}}\ [-\Delta_2^{[t+v]} V_m(\bar{n}, 0)] +$$

$$[\mu_1(n_1+1) - \mu_1(n_1)][-\delta_s V_m(\bar{n}+e_2, t+v)] + \mu_1(n_1)\Delta_3^s V_m(\bar{n}-e_1+e_2, t+v) +$$

$$[\mu_2(n_2+1) - \mu_2(n_2)]\Delta_2^s V_m(\bar{n}, t+v) + \mu_2(n_2)\Delta_3^s V_m(\bar{n}-e_2, t+v) +$$

$$[Q - h(t+s+v)\ 1_{\{n+1<N\}} - \mu_1(n_1+1) - \mu_2(n_2+1)]\ \Delta_3^s V_m(\bar{n}, t+v)\Big\} dv.$$

Here the $-\Delta_2 V_m(.)$ term is nonnegative but estimated from above by 1 as per induction hypothesis (25) for z=m. Hence, as before, by substituting this upper estimate, combining it with the first negative term which is exactly equal to its coefficient up to sign, applying the hypotheses: $\Delta_3^s V_m(.) \le 0$ and $-\delta_s V_m(.) \le 0$ and recalling (17) we conclude: $\Delta_3^s V_{m+1}(.) \le 0$.

Conversely, by deleting the nonnegative $-\Delta_2^s(.)$ term, applying $-\delta_s V_m(.) \ge -1$ and $\Delta_3^s V_m(.) \ge -1$ as per hypotheses and noting that all terms between braces {.} then sum up to -1 by virtue of (17), we obtain: $\Delta_3 V_{m+1}(.) \ge -1$. Inequality (26) is thus proven for z=m+1.

By induction the proof of the lemma is hereby completed. □

By result 3.1 and 3.2 we thus conclude:

Corollary 3.3 Assuming that h(t) is nondecreasing in t we have under (21):

(31) $|g - \bar{g}| \le 2\delta$

Remarks (Nondecreasing h(t))

(i) Note that only h(t) and not $\bar{h}(t)$ is required to be nondecreasing for corollary 3.3. For example, with h(t)=μ we can so investigate the effect of small deviations, as modeled by h(t), from an exponential input assumption.

(ii) The assumption of a nondecreasing failure rate h(t) is quite realistic. For instance, one can think of an arrival as representing a broken down component where the rate of a component to go down increases by its lifetime.

(iii) Extensions of result 3.2 relaxing that h(t) is not necessarily nondecreasing do seem possible along the same lines, but will be technically more complex. Particularly, a weighted mixture of decreasing and nondecreasing failure rates does seem possible.

References

[1] Adan, I.J.B.F. and Van der Wal, J. van der , (1987), "Monotonicity of the throughput of a closed queuing network in the number of jobs", To appear: *Opns. Res*.
[2] Adan, I.J.B.F. and Wal, J. van der, (1989), Monotonicity of the throughput in single server production and assembly networks with respect to the buffer sizes.; Proc. 1th International workshop on queueing systems with blocking, *North-Holland*, 345-356.
[3] Barbour, A. (1976), "Networks of queues and the method of stages", *Adv. Appl. Prob.* 8, 584-591.
[4] Cohen, J.W., (1980), "Sensitivity and insensitivity", Delft Progress Report 5, 159-173.
[5] Cao, X.R. (1985), "Convergence of Parameter Sensitivity Estimates in a Stochastic Experiment", *IEEE Trans. on Automatic Control*, Vol. AC-30, 834-843.
[6] Cao, X.R. (1987b), "First-Order Perturbation Analysis of a Single Multi-Class Finite Source Queue", *Performance Evaluation* Vol 7, 31-41.
[7] Cao, X.R. and Ho, Y.C. (1987),"Sensitivity Estimate and Optimization of Throughput in a Production Line with Blocking", *IEEE Trans. on AC*, Vol. 32, 959-967.
[8] Ho, Y.C., X.R. Cao, and Cassandras, C. (1983), "Infinitesimal and finite Perturbation Analysis for Queueing Networks", *Automatica*, 19, 4, 439-445.
[9] Ho, Y.C. and Li, S. (1988), "Extensions of Infinitesimal Perturbation Analysis", *IEEE Trans. on Automatic Control*, AC-33, 427-438.
[10] Glasserman, P. and Ho, Y.C. (Submitted), "Aggregation Approximations for Sensitivity Analysis of Multi-class Queueing Networks", *Performance Evaluation*.
[11] Hordijk, A. and Van Dijk, N.M. (1983), "Adjoint Process, Job-local-balance and Insensitivity of Stochastic Networks", *Bull. 44-th Session Int. Stat. Inst.*, Vol. 50, 776-788.
[12] Meyer, C.D. Jr., (1980), The condition of a finite Markov chain and perturbation bounds for the limiting probabilities, *SIAM J. Alg. Disc. Math.* 1, 273-283.
[13] Rohlicek, J.R. and Willsky, A.S. (1988), "The reduction of perturbed Markov generators: an algorithm exposing the role of transient states", *J.A.C.M.* 35, 675-696.
[14] Schassberger, R. (1978), "The insensitivity of Stationary Probabi-

lities in Networks of Queues", *Adv. Appl. Prob.* 10, 906-912.
[15] Schweitzer, P.J. (1968), "Perturbation theory and finite Markov chains", *J. Appl. Prob.* 5, 401-413.
[16] Seneta, E. (1967), "Finite approximations to infinite non-negative matrices", Proc. Cambridge, *Phil. Soc.* 63, 983-992.
[17] Seneta, E. (1968), "The principles of truncations in applied probability", *Comm. Math. Univ. Carolina* 9, 533-539.
[18] Seneta, E. (1980), "Non-negative matrices and Markov chains", Springer-Verlag.
[19] Shanthikumar, J.G. and Yao, D. (1987), "Stochastic monotonicity of the queue lenghts in closed queueing networks", *Oper. Res.* 35, 583-588.
[20] Shanthikumar, J.G. and Yao, D. (1988), "Throughput bounds for closed queueing networks with queue-dependent service rates", *Performance Evaluation* 9, 69-78.
[21] Shanthikumar, J.G. and Yao, D.D. (1988), "Monotonicity properties in cyclic queueing networks with finite buffers", Proceedings first international workshop on queueing networks with blocking, North-Carolina, may 1988.
[22] Stoyan, D. (1983), Comparison methods for queues and other stochastic models, Wiley, New York.
[23] Suri, R. (1985), "A concept of monotonicity and its characterization for closed queueing networks", *Opns. Res.* 33, 606-624.
[24] Suri, R. (1987), "Infinitesimal Perturbation Analysis for General Discrete Event Systems", *Journal of ACM*, Vol. 34, No. 3, 686-717.
[25] Suri, R. (1989), "Perturbation Analysis" The State of the Art and Research Issues Explained Via the GI/G/1 Queue", *Proceedings of the IEEE*, to appear.
[26] Tijms, H.C. (1986), *Stochastic Modelling and Analysis.* A computational approach. Wiley, New York.
[27] Tsoucas, P. and Walrand, J. (1983), Monotonicity of throughput in non-Markovian networks, To appear: *J. Appl. Probability.*
[28] Van Dijk, N.M. (1988), A formal proof for the insensitivity of simple bounds for finite multi-server non-exponential tandem queues based on monotonicity results, *Stochastic Procc. Appl.*, 27, 261-277.
[29] Van Dijk, N.M. (1988), "Perturbation theory for unbounded Markov reward processes with appplications to queueing", *Adv. Appl. Prob.* 20, 99-111.
[30] Van Dijk, N.M. (1989), "Simple performance bounds for non-product form queueing networks", Proceedings first international workshop on queueing networks with blocking, Norht-Holland, 1-18.
[31] Van Dijk, N.M., (1988), Simple bounds for queueing systems with breakdowns, *Performance Evaluation* 8, 117-128.
[32] Van Dijk, N.M., (1989), "A simple Throughput Bound for Large Closed Queueing Networks With Finite Capacities"; To appear: *Performance Evaluation*.
[33] Van Dijk, N.M. (1989),"On extended uniformization for non-exponential stochastic networks", to appear: *Adv. Appl. Prob.*.
[34] Van Dijk, N.M. and Lamond, B.F. (1988), "Bounds for the call congestion of finite single-server exponential tandem queues", *Opns. Res.* 36, 470-477.
[35] Van Dijk, N.M. and Puterman, M.L. (1988), "Perturbation theory for Markov reward processes with applications to queueing systems", *Adv. Appl. Prob.* 20, 79-98.
[36] Van Dijk, N.M. and Van der Wal, J. (1989), "Simple bounds and monotonicity results for multi-server exponential tandem queues", *Queueing Systems*, 4, 1-16.

[37] Van Marion, E.W.B., (1968), "Influence of holding time distributions or blocking probabilities of a grading ", *TELE* 20, 17-20.

[38] Whitt, W. (1981), Comparing counting processes and queues, *Adv. Appl. Prob.* 13, 207-220.

[39] Whitt, W. (1986), "Stochastic comparisons for non-Markov processes", *Math. Oper. Res.* 11 (4), 608-618.

[40] Whittle, P. (1985), "Partial Balance and Insensitivity", *Adv. Appl. Prob.* 22, 168-175.

1988-1	H. Visser	Austrian thinking on international economics
1988-2	A.H.Q.M. Merkies T. van der Meer	Theoretical foundations for the 3-C model
1988-3	H.J. Bierens J. Hartog	Nonlinear regression with discrete explanatory variables, with an application to the earnings function
1988-4	N.M.van Dijk	On Jackson's product form with 'jump-over' blocking
1988-5	N.M.van Dijk M.Rumsewicz	Networks of queues with service anticipating routing
1988-6	H.Linneman C.P.van Beers	Commodity Composition of Trade in Manufactures and South-South Trade Potential
1988-7	N.M.van Dijk	A LCFS finite buffer model with batch input and non-exponential sevices
1988-8	J.C.W.van Ommeren	Simple approximations for the batch-arrival $M^x/G/1$ queue
1988-9	H.C.Tijms	Algorithms and approximations for bath-arrival queues
1988-10	J.P.de Groot H.Clemens	Export Agriculture and Labour Market in Nicaragua
1988-11	H.Verbruggen J.Wuijts	Patterns of South-South trade in manufactures
1988-12	H.C.Tijms J.C.W.van Ommeren	Asymptotic analysis for buffer behaviour in communication systems
1988-13	N.M.van Dijk E.Smeitink	A non-exponential queueing system with batch servicing
1988-14	J.Rouwendal	Existence and uniqueness of stochastic price equilibria in heterogeneous markets
1988-15	H.Verbruggen	GSTP, the structure of protection and South-South trade in manufactures
1988-16	Mevr.H.Weijland Mevr.R Herweijer J.de Groot	Female participation in agriculture in the Dominican Republic
1988-17	N.M.van Dijk	Product Forms for Random Access Schemes
1988-18	A.H.Q.M.Merkies	Adaptive Forecasting with Hyperfilters
1988-19	J.Rouwendal	Specification and Estimation of a Logit Model for Housing Choice in the Netherlands
1988-20	J.C.W.van Ommeren R.D. Nobel	An elementary proof of a basic result for the GI/G/1 queue
1988-21	H.Kool	A Note on Consistent Estimation of Heterostedastic and Autocorrelated Covariance Matrices
1988-22	C.P.J.Burger	Risk Aversion and the Family Farm
1988-23	N.van Dijk I.F.Akyildiz	Networks with mixed processor sharing parallel queues and common pools
1988-24	D.J.F. Kamann P. Nijkamp	Technogenesis: Incubation and Diffusion
1988-25	P. Nijkamp L. van Wissen A. Rima	A Household Life Cycle Model For the Housing Market
1988-26	P. Nijkamp M. Sonis	Qualitative Impact Analysis For Dynamic Spatial Systems
1988-27	R. Janssen P. Nijkamp	Interactive Multicriteria Decision Support For Environmental Management
1988-28	J. Rouwendal	Stochastic Market Equilibria With Rationing and Limited Price Flexibility
1988-29	P. Nijkamp A. Reggiani	Theory of Chaos in a Space-Time Perspective
1988-30	P. Nijkamp J. Poot J. Rouwendal	R & D Policy in Space and Time
1988-31	P. Nijkamp F. Soeteman	Dynamics in Land Use Patterns Socio-Economic and Environmental Aspects of the Second Agricultural Land Use Revolution
1988-32	J. Rouwendal P. Nijkamp	Endogenous Production of R & D and Stable Economic Development
1988-33	J.A. Hartog E.Hinloopen P. Nijkamp	Multicriteria Methoden: Een gevoeligheidsanalyse aan de hand van de vestigingsplaatsproblematiek van kerncentrales
1988-34	R.van der Mark P. Nijkamp	The Development Potential of High Tech Forms in Backward Areas - A Case study for the Northern Part of The Netherlands
1988-35	E.R.K. Spoor J.W.B. Vermeulen	Principes en gebruik van Envisage
1988-36	C.Gorter P.Nijkamp P. Rietveld	The Duration of Unemployment: Stocks and Flows on Regional Labour Markets in the Netherlands
1988-37	M. Hofkes	Parametrization of simplicial algorithms with an application to an empirical general equilibrium model